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MATHEMATICS FOR QUALITY. LIVING AND NON-LIVING SYSTEMS

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ABSTRACT.

The traditional Differential Calculus often shows its limits when it tends to describe living Systems. These Systems in fact present such a richness of characteristics that are, in the majority of cases, much wider than the description capabilities of the usual differential equations.
This suggests we extend our concept of "derivative" in order to include at least fractional order derivatives. This simple step is already sufficient to give differential bases to the rules of Emergy Algebra (e.g., co-products, splits, etc.) and introduces a new class of functions (the "binary" functions) which are able to show how a "co-generated" product can be something more than the sum of the co-generating elements. This result, in turn, implies that the evaluation of a derivative can be determined in-dependently from the evaluation of its lower order derivatives. This enables us to introduce a different perspective in analyzing living Systems: we can follow their evolution from the very beginning, in their "rising", in their "incipient" act of being born, which is aptly described by the so-called "incipient" (or "spring") derivative.

Such an approach results not only as being more adequate to describe the specific processes typical of Life, but presents particularly interesting mathematical characteristics: i) the solutions to differential equations are always explicit (with enormous advantages of calculation) and, above all, ii) such solutions show a sort of "persistence of form" in the product generated with respect to the agents of the generating process. This mathematical aspect shows a profound analogy with genetic processes: every being retains genetic characteristics of its ancestors, even though, at the same time, it constitutes one new and sole being.

Even if the paper is especially finalized to the description of self-organizing living Systems, some specific examples devoted to non-living System will also be mentioned. In fact, what is much more surprising is that such an approach is even more valid (than the traditional one) to describe non-living Systems too (internal structure of the photon, precessions of Mercury, etc.).

1. INTRODUCTION. MATHEMATICS FOR SELF-ORGANIZING SYSTEMS

The Analysis of Complex Systems requires us to make use of

all those mathematical tools able to describe their time-space evolution as being even more strictly related to their *intimate specific structure*. In particular, if we want to analyze *self-organizing Systems* through the Maximum Em-Power Principle (Giannantoni, 2001c), this requires the knowledge of the mathematical structure of the *Emergy Source Terms*, in order to account for the increase in Quality corresponding to the different hierarchical structural levels of the analyzed Systems. At the same time it

would be desirable for all the rules of Emergy Algebra, in addition to their logical bases, to be explicitly derived on the basis of the generative processes which happen *inside* the System. As we will see, both such aspects are not only strictly related to each other, but they can also be thoroughly analyzed by means of a new mathematical approach: the *Intensive Fractional Differential Calculus*. In such a way both the intimate structure of Emergy Sources and the rules of Emergy Algebra can be contemporarily derived (under extremely general dynamic conditions) from a *unique* and innovative *mathematical* point of view.

However this (in general) cannot be considered sufficient in itself. In fact "generative" systems (such as living systems) require a more appropriate mathematical language to describe their *spring-activity*, in addition to the specific characteristics of the generated "product": the *persistence of the form* of its ancestors' genetic characteristics (although the generated being contemporarily constitutes something new). To this purpose, a generalization of the above-mentioned approach, suggested by one of its basic results (that is the independence of each derivative from the sequence of lower order derivatives), allows us to introduce the even more general differential perspective termed *Incipient Differential Calculus* (or *Prior Operator Differential Calculus*). The fundamental advantages of such a generalized mathematical approach will be shown through the analysis of some examples pertaining to self-organizing systems, by means of the solution of Riccati's and Abel's equations which can now be solved in *explicit terms*, both with constant and *variable* coefficients. We will also show how such an approach can allow us to face both old and new problems, in *Classical* and *Quantum Mechanics* respectively, by drawing some surprising and unexpected conclusions concerning non-living systems too.

Let us now start from the mathematical description of the *Emergy Source Terms* which appear in the mathematical formulation of the Maximum Em-Power Principle (Giannantoni, 2001c).

2. INTENSIVE FRACTIONAL DIFFERENTIAL CALCULUS AS A BASIC "LANGUAGE"

Let us consider the following linear differential equation of the first order with constant coefficients

$$\frac{dF(t)}{dt} + A \cdot \frac{d^{1/2}}{dt^{1/2}} F(t) + B \cdot F(t) = 0 \tag{2.1}$$

which contains a derivative of order Ω in addition to the traditional derivative of order one.

If we *take into account* that the fractional derivative of order Ω of the exponential function $e^{\alpha t}$ has *two distinct values* defined as follows (see Appendix 1)

$$\frac{d^{1/2}}{dt^{1/2}} e^{\alpha t} = \alpha^{\frac{1}{2}} e^{\alpha t} = \pm\sqrt{\alpha} \cdot e^{\alpha t} \tag{2.2}$$

it is possible to express the general solution to Eq. (2.1) by means of *two distinct* functions

$$f_1(t) = c_{11} e^{\alpha_{11}^2 t} + c_{12} e^{\alpha_{12}^2 t} \tag{2.3}$$

$$f_2(t) = c_{21} e^{\alpha_{21}^2 t} + c_{22} e^{\alpha_{22}^2 t} \tag{2.4}.$$

The function $f_1(t)$ is carried out by assuming the former structure in Eq. (2.2), that is , $+\sqrt{\alpha} \cdot e^{\alpha t}$ whereas $f_2(t)$ corresponds to the structure $-\sqrt{\alpha} \cdot e^{\alpha t}$, while the pertinent exponential coefficients are derived from the *two* associated characteristic equations (Giannantoni 2000b, 2001a,b). We may

now observe that, although the derivative $\frac{d^{1/2}}{dt^{1/2}}F(t)$ presents *two distinct values,* it conceptually constitutes *one sole entity*. In addition, it does not imply any particular assumption about the order of the sequence of the signs (+/- or -/+), so that the functions $f_1(t)$ and $f_2(t)$ should have *couples* of *corresponding exponents*, in principle interchangeable, according to the order of consideration of such signs. The general solution $F(t)$ may thus be written in a *compact form* as follows

$$F(t)=\begin{pmatrix} f_1(t) \\ f_2(t) \end{pmatrix}=\begin{pmatrix} c_{11} \\ c_{22} \end{pmatrix}\cdot e^{\begin{pmatrix} \alpha_{11} \\ \alpha_{22} \end{pmatrix}^2 t}+\begin{pmatrix} c_{12} \\ c_{21} \end{pmatrix}\cdot e^{\begin{pmatrix} \alpha_{12} \\ \alpha_{21} \end{pmatrix}^2 t} \qquad (2.5),$$

where, by convention, the *upper* exponents refer to the function (corresponding to the choice of the sign "plus") whereas the *lower* exponents refer to the function (corresponding to the choice of the sign "minus"). On the basis of Eq. (2.5), the *comprehensive* solution may be named as a "binary" function, not only because it is made up of *two* distinct functions, but also (and especially) because the two components are so strictly related that they form *one sole entity*.
In spite of this consideration, the initial conditions for the function (solution to Eq. (2.1)) can be given, *in principle*, without any particular restriction, for instance as follows

$$F(0)=\begin{pmatrix} f_{10} \\ f_{20} \end{pmatrix} \qquad (2.6) \qquad \text{and} \qquad F^{(1/2)}(0)=\begin{pmatrix} f_{10}^{(1/2)} \\ f_{20}^{(1/2)} \end{pmatrix} \qquad (2.7),$$

conditions which only require the prefixed up/down convention of signs to be respected.

3. EMERGY SOURCE TERMS AND "BY-PRODUCTS" IN THE LIGHT OF THE RACTIONAL CALCULUS

An equivalent mathematical model (chosen from several possible) for a ´co-production process[a], in *steady state conditions*, has already been analyzed in Giannantoni (2000a). In that case (see Fig.1) we showed that the Emergy Flow Balance could be written as follows

$$\dot{E}m(u)+\Phi(u)=\dot{E}m(y_1)+\dot{E}m(y_2) \qquad (3.1)$$

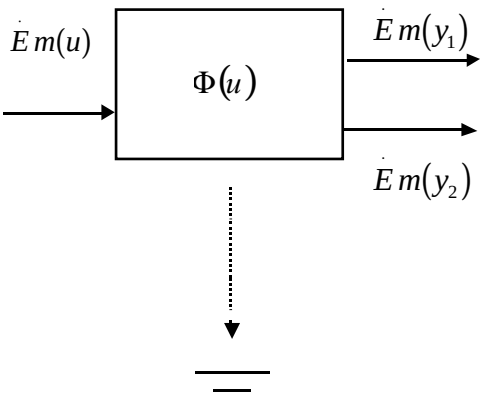


Figure. 1*- Equivalent mathematical model for a co-production process*

where

$$\dot{E}m(y_1) = \dot{E}m(y_2) \qquad (3.2) \qquad \text{and} \quad \Phi(u) = \dot{E}m(u) \qquad (3.3),$$

whereas now, in *variable conditions*, we have to take the Emergy Accumulation Term into account, so that Eq. (3.1) becomes

$$\dot{E}m(u) + \Phi(u) = \frac{\partial}{\partial t} A_D + \dot{E}m(y_1) + \dot{E}m(y_2) \qquad (3.4)$$

where the term $\partial A_D / \partial t$ represents the "local" variation (in the Eulerian sense) of the Accumulated Emergy on behalf of the considered System.
In writing the previous equation we simply assume that the analyzed System represents a co-productive process only on the basis of the *relational structure* of its pertinent outputs, that is *without considering its internal productive structure*, which, in any case, does not "appear" explicitly in Eq. (3.4). So that, in order to reach a possible mathematical description of its internal productive structure, we may now compare Eq. (3.4) with a *fractional differential equation*, written in a unique variable (already thought of as a *flow*, for simplicity of notation), whose homogeneous part is similar to Eq. (2.1), that is

$$C \cdot \frac{dEm(t)}{dt} + A \cdot \frac{d^{1/2}}{dt^{1/2}} Em(t) + B \cdot Em(t) = Em[u(t)] \qquad (3.5).$$

As a first result it is then easy to show that, if we assume the output Emergy Flow to be proportional to the accumulated Emergy (as is usual in physical and biological systems)

$$Em = k \cdot A_D \qquad (3.6)$$

it follows that, for $C = k \qquad A = 1$ and $B = 2$ (3.7),
Eq. (3.5) presents a solution in the variable which is, in principle, a *binary* function

$$Em(t) = \begin{pmatrix} Em(y_1(t)) \\ Em(y_2(t)) \end{pmatrix} \qquad (3.8)$$

whose "components" are the *two* un-known output functions. We will now show in detail, through the analysis of the explicit solution, that the definition of "co-products" *directly* derives from the *inner structure* of the System and that the "Source Term" is *strictly* related to the *derivative* of order 1/2.[1]
The explicit solution of Eq. (3.5) is formally given by (see Giannantoni, 1995)

$$Em(t) = \varphi(t, D)^{-1} \cdot [Em(u(t)) + \sum_{k=0}^{1} Em_0^{\left(\frac{k}{2}\right)} D^{\frac{1-k}{2}} \delta(t)] \qquad (3.9)$$

where

$$\varphi(t, D) = C \cdot (D^{\frac{1}{2}})^2 + A \cdot D^{\frac{1}{2}} + B \qquad (3.10)$$

is a second order polynomial operator in the basic operator $D^{\frac{1}{2}}$, *while* D *is a differential operator simply defined as follows (by means of the Dirac delta function* $\delta(t)$*)(ib.)*

$$Df(t) = \frac{d}{dt} f(t) - f(0)\delta(t) \tag{3.11}$$

which allows us to directly introduce the associated initial conditions (generally considered separately) into the formal solution (3.9). Such a solution can also be given in explicit terms

$$Em(t) = \int_0^t k(t,\tau) \cdot [Em(u(t)) + \sum_{k=0}^{1} Em_0^{\left(\frac{k}{2}\right)} D^{\frac{1-k}{2}} \delta(t)] \cdot d\tau \tag{3.12}$$

where

$k(t,\tau)$ is the *solving kernel* of Eq. (3.5), which always has an*explicit form* (see par. 6).

The formal structure of Eq. (3.9) allows us to draw the following main conclusions:

i) the operator $\varphi(t,D)^{-1}$ is the one which characterizes *the generative action* of the System;

ii) in the absence of any *external* input that acts as a forcing action (that is $Em\big(u(t)\big) = 0$) and specific initial conditions (different from zero), it represents only a generative *potentiality*;

iii) in the case of given initial conditions (and in the absence of any *external* forcing action), it defines the *specific* evolution of the System, which expresses its *intrinsic* potentialities by taking into account the actually given boundary conditions;

iv) even in the presence of a given *external* forcing action $Em\big(u(t)\big)$, the operator $\varphi(t,D)^{-1}$ continues to express its specific *originality* by "filtering" such an external occasional condition and by modulating it in such a way as to transform it into something new. In fact, even if the input Emergy $Em\big(u(t)\big)$ is a "monadic" function (see par. 5), the latter is *trans-formed* by $\varphi(t,D)^{-1}$ into a *binary function.*

This clearly illustrates, through this new mathematical "language", the well-known experimental evidence that the output

diversification is often related to an increase in the structural level of the System (e.g., let us think of biodiversity).

The same considerations can also be drawn by starting from the explicit solution (3.12).

The combined consideration of Eqs. (3.9) and (3.12) also allows us to clarify (in mathematical terms) the fundamental distinction between *effective causes*, *concomitant* (or accidental) *circumstances* and associated *external conditions* already dealt with in reference to the mathematical formulation of the Maximum Em-Power Principle (Giannantoni, 2001c).

Moreover, the direct comparison between Eqs. (3.4) and (3.5), yields

$$\Phi(u) = -A \cdot \frac{d^{1/2}}{dt^{1/2}} Em(t) \tag{3.13}$$

which shows how the Source Term $\Phi\big(u\big)$ can be *directly related* to the term containing the derivative of order Ω. Such a result will become much more meaningful in the light of the *Prior Operator Differential Approach* (introduced later on, in par. 6), according to which the operator $\frac{d^{1/2}}{dt^{1/2}}$ is the one which "generates" (as a *prior* operator) the "bifurcation" of $Em(t)$. For the moment it is sufficient to point out that the fractional derivative of order Ω is responsible for such a *new form* of "bifurcation" (understood as a "generation" of a "binary" function), which well illustrates one of the fundamental *basic* processes

through which a System can self-organize its structure and yield *co-products* of a higher physical "Quality". Such a "Quality", in turn, can be *hierarchically* associated to the higher *ordinality* (binary, ternary, and so on) of the output multiple functions of the System.

4. IRREDUCIBILITY OF THE PROBLEM TO ORDINARY DERIVATIVES. "SPLITS" AS "DUAL" FUNCTIONS

Alternatively, one could think of interpreting the given "co-production" process in terms of *two distinct* output functions which satisfy the *same* differential equation, even if they are subjected to *different* initial conditions.

Let us consider, for instance, the following equation which, potentially, could be apt to describe the process under consideration:

$$C \cdot \frac{d^2}{dt^2}\begin{bmatrix} Em_1(t) \\ Em_2(t) \end{bmatrix} + B \cdot \frac{d}{dt}\begin{bmatrix} Em_1(t) \\ Em_2(t) \end{bmatrix} + A \cdot \begin{bmatrix} Em_1(t) \\ Em_2(t) \end{bmatrix} = Em(u(t)) \qquad (4.1)$$

with the following well-posed initial conditions

$$\begin{bmatrix} Em_1(0) \\ Em_2(0) \end{bmatrix} = \begin{bmatrix} Em_{10} \\ Em_{20} \end{bmatrix} \quad (4.2) \quad , \quad \begin{bmatrix} Em_1'(0) \\ Em_2'(0) \end{bmatrix} = \begin{bmatrix} Em_{10}' \\ Em_{20}' \end{bmatrix} \qquad (4.3).$$

The general solution of the associated homogeneous equation is then given by

$$\begin{bmatrix} Em_1(t) \\ Em_2(t) \end{bmatrix} = \begin{bmatrix} C_{11} \\ C_{22} \end{bmatrix} \cdot e^{\lambda_1 t} + \begin{bmatrix} C_{12} \\ C_{21} \end{bmatrix} \cdot e^{\lambda_2 t} \qquad (4.4)$$

where the exponents and are the solutions of the *unique* characteristic equation and the coefficients are defined by the conditions (4.2) and (4.3).

At a first glance such a solution could seem "similar" to the expression given by Eq. (2.5) and this fact would consequently suggest we analyze the possibility of reinterpreting the behavior, previously described in terms of *one fractional* differential equation, as being possibly describable by this differential equations of *integer* order. If we thus impose identical initial conditions for both problems, that is

$$F(0) = \begin{pmatrix} Em_{10} \\ Em_{20} \end{pmatrix} \qquad (4.5) \quad , \quad F^{(1/2)}(0) = \begin{pmatrix} Em_{10}' \\ Em_{20}' \end{pmatrix} \qquad (4.6),$$

it could *seem* possible for us to interpret the function expressed by Eq. (2.5) as a compact solution to Eq. (4.1), with the pertinent initial conditions (4.2) and (4.3). That is it might be thought possible for us to write

$$F(t) = \begin{pmatrix} f_1(t) \\ f_2(t) \end{pmatrix} = \begin{bmatrix} Em_1(t) \\ Em_2(t) \end{bmatrix} \qquad (4.7).$$

In reality there is a profound *conceptual* difference between the last member of Eq. (4.7) and the function , so that such a particular relationship holds only *under very particular specific conditions*. In fact, the "couple" of functions and constitutes a *mathematical entity* which is something *more* than a *simple vector* made up of the two components $Em_1(t)$ and $Em_2(t)$. On the one hand, in fact, $Em_1(t)$ and $Em_2(t)$ satisfy the same Eq. (4.1) *independently*. The corresponding initial conditions may or may not be related, and this implies that it is possible to think that the two functions are in reality solutions to *two distinct* differential problems (Giannantoni 2000b, 2001a,b). On the other hand the function is the *sole* solution to the *unique* differential problem represented by Eq. (2.1) (completed by its associated

initial conditions): its sub-division in *two* distinct equations is just a *preliminary* and *artificial* procedure in order to evaluate the two functions and, according to an *arbitrary* prefixed sequence of signs (+/- or -/+); such a distinction in fact is afterwards *re-composed* through the assumption of the compact form (2.5) which expresses the *sole*, *comprehensive and general solution*. Moreover, the fact that the derivative of order Ω is *one sole entity* (although giving rise to two distinct results) does not imply, in particular, that the initial conditions (2.6) and (2.7) necessarily specialize as follows

$$F(0) = \begin{pmatrix} f_0 \\ f_0 \end{pmatrix} \qquad (4.8) \qquad\qquad F^{(1/2)}(0) = \begin{pmatrix} +f_0^{(1/2)} \\ -f_0^{(1/2)} \end{pmatrix} \qquad (4.9).$$

This is because the definition of a "binary" function depends *only* on the type of the unique differential equation. More precisely, *on its basic fractional derivative of order 1/2*. In fact, even if such previous conditions are not satisfied, the solution of the fractional differential equation does not lose its *unity* (as a binary function). It might "degenerate" into a "dual" function, that is a function made up of two independent "monadic"[2] functions (extrinsically related, as in the case of a vector), *only* if we *decide beforehand* to analyze the trend of such (supposedly independent) solutions *exclusively in terms of ordinary derivatives* (that is on the basis of a particular perspective, *preliminarily* chosen, which implicitly *excludes* other possibilities). In fact: *a* fractional *differential problem is never reducible to an* ordinary *differential problem without losing its specific intrinsic characteristics*.

This also implies that the behavior described by one

fractional differential equation *univocally* characterizes (and consequently *defines*) a co-production process, whereas a traditional *vector* differential equation is only able to describe (by means of its solutions as "dual" functions) a *split* process. Consequently, this implies that the two distinct processes (co-production and split, respectively) *can never be confused*, because their intrinsic definition is *now* based on the *intimate structure* of each System, which is unequivocally described in differential terms, profoundly (because essentially) different: *fractional* basic derivatives and *integer* basic derivatives, respectively.

5. REDUCIBILITY OF *NON-LINEAR* PROBLEMS TO *LINEAR* DIFFERENTIAL EQUATIONS OF *FRACTIONAL* ORDER

If we now consider the same equation (2.1) with the following initial conditions

$$F(0) = \begin{pmatrix} f_0 \\ f_0 \end{pmatrix} \qquad (5.1) \qquad F^{(1/2)}(0) = \begin{pmatrix} +f_0^{(1/2)} H(t) \\ -f_0^{(1/2)} H(t) \end{pmatrix} \qquad (5.2)$$

where is the Heaviside function, we obtain the solution still structured in the form (2.5) where the pertinent "binomial" coefficients are given by

$$\begin{pmatrix} c_{11} \\ c_{22} \end{pmatrix} = \begin{pmatrix} \dfrac{f_0\alpha_{12} - f_0^{(1/2)} H(t)}{\alpha_{12} - \alpha_{11}} \\ \dfrac{f_0\alpha_{12} + f_0^{(1/2)} H(t)}{\alpha_{12} - \alpha_{11}} \end{pmatrix} \qquad (5.3) \qquad \begin{pmatrix} c_{12} \\ c_{21} \end{pmatrix} = \begin{pmatrix} \dfrac{-f_0\alpha_{11} + f_0^{(1/2)} H(t)}{\alpha_{12} - \alpha_{11}} \\ \dfrac{-f_0\alpha_{11} - f_0^{(1/2)} H(t)}{\alpha_{12} - \alpha_{11}} \end{pmatrix} \qquad (5.4).$$

Such a solution corresponds to a "binary" function whose *branches*, although *coinciding* for , give rise to a "bifurcation" for (see Giannantoni 2000b, 2001a,b).

Such a *bifurcation* should *not* be understood as being made up of two distinct and independent functions, but as a *unique* binary function (in the sense previously specified) which always remains the *same mathematical entity* both if its "branches" coincide or diverge.

This simple example shows how the new concept of fractional derivative and the associated new class of functions (the "binary" functions) enable us to re-interpret a bifurcation (generally associated to a *non-linear* differential problem) as a solution of a *linear fractional* differential equation in the basic derivative of order 1/2. This also points out (from a more general point of view) the interest of such an approach in Emergy Analysis of Complex Systems. It is sufficient in fact to think of those wide classes of equations that can be derived through a *linear* combination of an *increasing number and order of different fractional basic derivatives*, contemporarily present in the same equation (or systems of equations), which may be at the same time characterized by an *arbitrary number* of "modes"[3]. This consequently allows us to assert that: "a wide class of phenomena whose *behavior is generally thought of as "non-linear" could* in actual fact *be linear*, *if analyzed in terms of intensive fractional differential equations*".

Let us now consider the above-mentioned *generalization* of the previous approach, which is fundamentally based on the *direct priority* of differential operators, a methodology which is particularly indicated when describing *living* Systems.

6. *GENERATIVE* SYSTEMS AND *PERSISTENCE* OF FORM

The characteristics of *living* Systems as *generative* Systems stimulated us to look for more adequate mathematical concepts able to describe their peculiar *generative* processes. This led us to the *new concept of derivative* we are going to present, elaborated in turn as a generalization of the *fractional* derivative previously shown (and synthesized in Appendix 1).

Such a different perspective starts from the consideration of the fact that the traditional definition of the derivative of a function

$f(t)$ given in Mathematical Analysis

$$\lim_{\Delta t \to 0} \frac{\Delta}{\Delta t} f(t) \tag{6.1}$$

may be considered as being an *a posteriori* definition (e.g., let us think of the definition of velocity). In fact, although it is usually *read* from left to right, it is vice versa *interpreted* from right to left. In other words its meaning is based on a *reverse priority of the order* of the three elements that constitute its definition: i) the concept of function (which is assumed to be a *primary* concept); ii) the incremental ratio (of the *supposedly* known function); iii) the operation of limit (*referred to the result* of the previous two steps).

Now we may ask: what happens if we interpret the *sequence of symbols* in expression (6.1) according to the *same order as they are written* (that is from left to right)?

Such a *direct* perspective gives rise to a *new concept of derivative* (indicated by $\frac{\tilde{d}}{\tilde{d}t}$ and defined in Appendix 2) which can be named "incipient" (or "spring") because of some special characteristics that will be illustrated through the derivative of the exponential function $e^{\varphi(t)}$, which now gives

$$\frac{\tilde{d}^n}{\tilde{d}t^n} e^{\varphi(t)} = \left(\frac{\tilde{d}}{\tilde{d}t} \varphi(t) \right)^{\tilde{n}} \cdot e^{\varphi(t)} = \left(\tilde{\varphi}' \right)^n \cdot e^{\varphi(t)} \tag{6.2}.$$

Such a result is always *formally* different from the one obtainable through traditional ordinary derivatives, even when both results coincide *numerically* (that is, for any order derivative, if

$\varphi(t) = \alpha t + \beta$, otherwise, if $\varphi(t)$ is a non-linear function, only in the case of first-order derivative). Consequently the adopted symbology reminds us of the main differences: i) the resulting expression refers to a *virtual* evolution, which may also become a *real* evolution, *but only* in dependence on particular boundary conditions; ii) the comprehensive structure of Eq. (6.2) reminds us that the obtained result is due to a "generating process", the virtual (evolutive) possibilities of which are delineated in terms of its intrinsic *genetic* characteristics $(\tilde{\varphi}')^n$, which are essentially due to *both* the *generator* $\frac{\tilde{d}^n}{\tilde{d}t^n}$ (understood as a *prior* "operator") and the "fertile" co-operation of the considered function' $e^{\alpha t}$; iii) thus the final result represents an evolutive modality which is *completely new* with respect to the original function: it is not seen now as a "necessary" consequence (as in the case of operators interpreted *a posteriori*) but, because of the *a priori* interpretation of operators, it is conceived as an "adherent" consequence of its "generation" modalities: all the various functions resulting from the "generating process" represented by Eq. (6.2), for $n \lfloor N$, are a similar to *harmonic evolutions* which are in "resonance" (as in a "musical chord") with the original function and at the same time with each other.

7. EXAMPLES OF APPLICATION TO *LIVING* AND *NON-LIVING* SYSTEMS

We may start from the consideration that some very simple *self-organization* processes pertaining to *Living Systems* can be described by the *simplest* non-linear equation represented by Riccati's equation. H. Odum devoted many pages to the description of such processes (e.g., Odum 1994a, chap. 9). What it is here worth pointing out is that the most general Riccati equation with *variable coefficients*, if understood now in terms of incipient derivatives,

$$\frac{\tilde{d}f}{\tilde{d}t} + Q(t)f(t) + R(t)f^2(t) = P(t) \tag{7.1}$$

always has an *explicit* solution. This equation, in fact, through the substitution (Davis, 1960)

$$y(t) = \frac{1}{f(t)R(t)} \cdot \frac{\tilde{d}f}{\tilde{d}t} \tag{7.2},$$

reduces to a second-order linear differential equation with variable coefficients

$$R\frac{\tilde{d}^2}{\tilde{d}t^2}y(t) - (\tilde{R}' - QR)\frac{\tilde{d}}{\tilde{d}t}y(t) - PR^2 y(t) = 0 \tag{7.3}.$$

The advantage of the *incipient* derivatives is due to the fact the *all* the linear differential equations, of *any integer or fractional* order, now have an *explicit solution*. In particular, with reference to Eq. (7.3), the *incipient derivative* is now able to solve the secular problem of giving an *explicit solution* to a second order linear differential equation with variable coefficients

$$\frac{\tilde{d}^2}{\tilde{d}t^2}\tilde{f}(t) + a_1(t)\frac{\tilde{d}}{\tilde{d}t}\tilde{f}(t) + a_0(t)\tilde{f}(t) = 0 \tag{7.4}$$

with its well-posed initial conditions

$$\tilde{f}^{(k)}(0) = \tilde{f}_{k0} \quad \text{for} \qquad k = 0,1 \tag{7.5}$$

by means of *finite terms and quadratures*. If we remember in fact the property of the "incipient" derivative of an exponential function such as $e^{\varphi(t)}$ (see Eq. (A2.4) in Appendix 2) and substitute such an expression into Eq. (7.4), we obtain

$$\left(\tilde{\varphi}'(t)\right)^2 + a_1(t)\left(\tilde{\varphi}'(t)\right) + a_0(t) = 0 \tag{7.6}$$

which is an *algebraic* equation (with variable coefficients) in the un-known function $\tilde{\varphi}'(t)$.

If $\tilde{\alpha}_i(t)$ $(i = 1,2)$ are the two solutions to Eq. (7.6), we can easily obtain two independent solutions $\varphi_i(t)$ to the initial equation (7.4) through the following "incipient" integrals[4]

$$\tilde{\varphi}_i(t) = \int_0^{\tilde{1}t} \tilde{\alpha}_i(u)\,\tilde{d}u \qquad (i = 1,2) \tag{7.7}$$

so that the general solution to Eq. (7.4) is given by

$$f(t) = \sum_{i=1}^{2} c_i e^{\int_0^{\tilde{1}t} \tilde{\alpha}_i(u)\,\tilde{d}u} \tag{7.8}$$

where the constant coefficients c_i are defined by means of the initial conditions (7.5).

If, differently, the solution $\tilde{\alpha}_1(t)$ has a multiplicity $\nu_1 = 2$, an additional independent "incipient" integral is given by

$$y_2(t) = \int_0^{\tilde{1}t} e^{\int_\xi^{\tilde{1}t} \tilde{\alpha}_i(u)\,\tilde{d}u}\,\tilde{d}\xi \tag{7.9}.$$

Such *explicit* solutions enable us to extend the results achieved for *basic self-organizing* systems, describable by a Riccati equation with *constant* coefficients to more Complex *Living* Systems whose behavior is described by Abel's *non-linear* equations (e.g. Odum 1994a, p.151) of *any order*, with *variable* coefficients. Such equations, as it is well-known, are structured as

$$\frac{\tilde{d}}{\tilde{d}t}\tilde{f}(t) = F[t, \tilde{f}(t)] = \sum_{k=0}^{n} A_k(t)\cdot[\tilde{f}(t)]^k \tag{7.10}$$

where $F[t, \tilde{f}(t)]$ is a polynomial of integer order n in $\tilde{f}(t)$. Eq. (7.10) in fact constitutes a generalization of Riccati's equation (with variable coefficients) and it can *now* be analogously solved in explicit terms, if previously reduced to a linear equation through an appropriate transformation of variable (Davis, 1960).

What is important to underline is that the solutions obtained by means of this differential approach *coincide* exactly with the traditional solutions to differential equations *when these are available in finite terms and quadratures*, that is: in the case of linear differential equations (of *any* integer order) with *constant coefficients* and *first order* equations with *variable coefficients*. *In all the other cases* the

traditional solutions which are obtained, for instance through expansion series, gradually *differ* from the explicit solutions obtained in terms of incipient derivative. In other words *traditional solutions present a sort of* "*drift*", which is even more marked according to the increasing order of the involved derivatives.

As an example of such a "shifting" we can mention a well-known problem of Celestial Mechanics: the *precession of the planets* (especially Mercury). In fact, on the basis of Newton's Laws, the *classical* variation ($\Delta\varphi_c$) of the angular anomaly per each revolution is given by the following expression (Landau and Lifchitz, 1969)

$$\Delta\varphi_c = 2 \cdot \int_{r_{\min}}^{r_{\max}} \frac{(M/r^2)dr}{\sqrt{2m(E-U)-(M/r)^2}} \tag{7.11}$$

which, however, vanishes after a number of $2\pi m/n$ revolutions (where m and n are integer numbers) because the central force field is characterized by a potential energy which is proportional to $1/r$ (ib.). If vice versa we re-interpret the basic differential equation of the angular anomaly in terms of *incipient* derivatives

$$\frac{\tilde{d}^2}{\tilde{d}t^2}\tilde{\varphi} + 2 \cdot \left(\frac{\tilde{d}}{\tilde{d}t}\tilde{\varphi}\right)^2 = 0 \tag{7.12},$$

we may search for an approximate solution to non-linear Eq. (7.12) through two successive linearization processes: the first one based on the approximation of the *sole* first-order "incipient" derivative, the second one *also* based on the approximation of the second-order "incipient" derivative. The corresponding solutions to the two *linearized* versions of Eq. (7.12) are given respectively by the following expansion series

$$\Delta\tilde{\varphi}_1 = \Delta\varphi_c - \frac{1}{2}(\Delta\varphi_c)^2 + \frac{1}{6}(\Delta\varphi_c)^3 + ... \tag{7.13}$$

$$\Delta\tilde{\varphi}_2 = \Delta\varphi_c - \left(\frac{\Delta\varphi_c}{2}\right)^2 + \frac{1}{3}\left(\frac{\Delta\varphi_c}{2}\right)^3 + ... \tag{7.14}.$$

Consequently, an estimation of the real *secular* precession can be given respectively by

$$\Delta\varphi_{\text{sec},1} = \Delta\tilde{\varphi}_1 - \Delta\phi_c = \frac{1}{2}(\Delta\varphi_c)^2 - \frac{1}{6}(\Delta\varphi_c)^3 + ... = 86.12" \tag{7.15}$$

and

$$\Delta\varphi_{\text{sec},2} = \Delta\tilde{\varphi}_2 - \Delta\phi_c = \left(\frac{\Delta\varphi_c}{2}\right)^2 - \frac{1}{3}\left(\frac{\Delta\varphi_c}{2}\right)^3 + ... = 43.06" \tag{7.16}$$

The latter result, in particular, which is in almost perfect agreement with the most recent available data[5], has still to be considered as an approximation, because it is both the result of a *linearization* process and in addition its value presents a rather marked *sensitivity* to the initial data. In any case whatsoever the shown solution process based on the *incipient* derivatives suggests a profound *reflection* not only concerning Classical and Relativistic Mechanics and their reciprocal relationship, but also the reciprocal influence between *physical* "Models" and *mathematical* "Methods". In fact, the obtained results indicate a well-defined line of tendency: Newton Laws could be *still* considered as being substantially *adequate* to describe such an effect which has been (up to now) inexplicable in terms of Classical Mechanics (in fact it was explained by Relativity, but only on the basis of a *different physical* model). This implies that the *defect could be attributed more to the mathematical language adopted rather than to an intrinsic deficiency in Newton's Laws*. At the same time it is worth pointing out that Eq. (7.12), which has been assumed as a basic starting point, has to be considered as being only a preliminary approximation of the

physical reality. In fact, the "binary" functions introduced in par. 2, suggest we describe the whole System made up of Sun and Mercury as a "binary" System. Such a *physical model* should be thus more appropriate to describe (still in terms of *a priori fractional* derivatives) the phenomenon under consideration, not only from a quantitative point of view, but from the Quality point of view, especially because of the associated required modifications of the concepts of *space* and *time* which are consequently involved.

The above-mentioned "drift" effects are also fundamental to re-interpret the well-known mathematical aspect named as "deterministic chaos". For the sake of brevity, this will only be mentioned here. What, on the contrary, is worth pointing out is that the introduction of the incipient derivatives does not imply a deterministic perspective but, on the contrary, suggests a new perspective: *adherentism*, understood as a dynamic evolution in which every "product" generated, besides representing a substantial novelty, is always "faithful" to the presuppositions of its generation, that is it is always in *consonance* with its "ancestors' genetic characteristics" (see Giannantoni, 2001c).

As a concluding example, we can here recall the preliminary model of the "creation of a positron-electron couple" presented in Giannantoni (2000b, 2001a,b) and there analyzed in terms of *intensive* fractional derivatives. In fact, if the *same* basic equations that describe the model are now re-interpreted in terms of fractional *spring*-derivatives, the considered model is able to show how such a kind of "creation" should be understood: not as an *ex nihilo* creation, but as a "creative" (in the sense of "original") *response* of the gamma ray to the external concomitant circumstance (in this case, an elastic interaction), in order to maintain its *binary entity*, in spite of the adverse external conditions. The gamma ray, by "filtering" and "modulating" such an external input, transforms into something else which is phenomenologically new: a positron and an electron which, although distinguishable between themselves, at the same time always remain describable by the *same* binary function. The *persistence of the form* of the binary function which contemporarily represents both of them, indicates that they cannot even be considered as being already present *as such* in the gamma ray, but as being generated as a consequence of a *re-organization* of its intimate structure.

8. CONCLUSIONS

The previous considerations and corresponding explicative examples should have clearly shown how the *basic* presuppositions of the traditional mathematical approach to *living*

Systems is rather restrictive and somewhat reductive. In particular, the consideration of differential equations of *only* integer order (generally adopted to describe such Systems) is rather limiting with respect to the much wider variety of their biodiversity characteristics. Vice versa, the introduction of an *intensive* concept of *fractional* derivative and, above all, the concept of *incipient* differential equations (of *integer* and *fractional* order) can represent a valid approach to describe and analyze the *spring-dynamism* of such Systems. In fact, we have shown that: i) *fractional* derivatives generate a new class of functions ("binary", "ternary" functions, and so on) that are able to describe the new reality generated by a given process as being *one sole entity*; ii) they enable us to express, in explicit form, the Emergy Source Terms which appear in the mathematical formulation of the Maximum Em-Power Principle; iii) they also enable us to distinguish *unequivocally* "by-products" from "splits", simply on the basis of the differential operators that describe the intimate structure of the concerned process; iv) to *re-interpret* a simple (or multiple) bifurcation, generally thought of as being described by a *non-linear* differential equation of *integer* order, as a solution to a *linear* differential equation in the *fractional* derivative of order Ω (or higher); v) fractional derivatives also suggest that more general differential equations, structured as a *linear* combination of an ever *increasing number of different fractional basic derivatives*, in turn characterized by an *arbitrary number* of "modes", may be able to describe wide classes of phenomena whose behavior is generally *thought of* as being *"non-linear"*; vi) if then such equations are interpreted in terms of *incipient* derivatives, they enable us to analyze in explicit terms the dynamic behavior of more complex *self-organizing* Systems described by Riccati's and Abel's non-

linear equations; vii) because of their different conceptual definition, incipient (or "spring") derivatives allow us to follow the evolution of living Systems from their very beginning, in their "rising", in their "incipient" act of being born; viii) at the same time they are able to describe the ever present *persistence*, in the "product" generated, of its ancestors' *genetic* characteristics; ix) in addition, the general structure of the differential equations allows us to identify and distinguish *effective causes*, *concomitant* (or occasional) *causes* and *external given conditions*, so that it is possible for us to avoid the so-called *aphorism of the false cause*; x) moreover such differential equations *always* have an *explicit* solution in *finite terms and quadratures*; xi) such solutions are thus not affected by that "drift" phenomenon (due to the traditional *a posteriori* derivatives) which is always present even in linear equations and that is (mainly) responsible for the so called "deterministic chaos"; xii) consequently, this even allows us to answer more clearly well-known problems of Classical Mechanics (see Mercury's precessions); xiii) and to delineate preliminary solutions to unsolved aspects of Quantum Mechanics (the internal structure of the photon).

What in fact is much more surprising in (and in favor of) such an approach is the fact that it is not only more adequate to describe *living* Systems, but even more valid to describe *non-living* Systems too.

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Footnotes

[1] The following explanation, which is
always valid in terms of "incipient" derivatives (see par. 6), is *also* valid in terms of *intensive* fractional derivatives, because the linear equations here considered are with *constant* coefficients, and their solutions are consequently based on the exponential function as a "hinge" function. Such an explanation, however, here introduced only for the sake of simplicity and clarity (and not for generality), should be already thought of in terms of *incipient* derivatives, and consequently as a generalized explanation.

[2] That is each one can be thought of as a solution of an independent *ordinary* differential equation.

[3] The number of *modes* is defined as the ratio between the order of the differential equation and the order of the basic fractional derivative (Giannantoni 2000b, 2001a).

[4] Such "incipient" integrals will not analyzed here for the sake of space. What it is vice versa worth pointing out is their property of "absolute locality" (or absolute *absence* of "non-locality"), which is closely related to the "absence of drift" in the explicit solutions mentioned few lines ahead.

[5] Astronomical measurements give 42.6''± 0.9'' per century. The value predicted by Relativity Theory is 43.0'' per century (Landau and Lifchitz, 1969).

APPENDIX 1. INTENSIVE FRACTIONAL DERIVATIVE

Its definition is given (see Giannantoni 2000b, 2001a) through a *decompositon-recomposition* procedure initially based on an extension of the traditional definition of derivative of order η to any rational number $q \in Q$:

$$\frac{d^q}{dt^q} f(t) = \lim_{\Delta t \to 0} \frac{(\delta_{\Delta t} - 1)^q}{\Delta t^q} f(t) \tag{A1.1}$$

where the Newton expansion of the second side introduces a *linear* operator $\delta^n_{\Delta t}$ such as

$$\delta^n_{\Delta t} f(t) = f(t + n\Delta t) \tag{A1.2}.$$

Such a definition, if applied to the function $f(t) = e^{\alpha t}$, gives

$$\frac{d^q e^{\alpha t}}{dt^q} = \lim_{\Delta t \to 0} \frac{\sum_{k=0}^{\infty} (-1)^k \binom{q}{k} \delta^{q-k}_{\Delta t}\left(e^{\alpha t}\right)}{\Delta t^q} \cdot e^{\alpha t} = \lim_{\Delta t \to 0} \frac{\sum_{k=0}^{\infty} (-1)^k \binom{q}{k} e^{\alpha[t+(q-k)\Delta t]}}{\Delta t^q} =$$

$$= \lim_{\Delta t \to 0} \frac{\sum_{k=0}^{\infty} (-1)^k \binom{q}{k} \left(e^{\alpha \Delta t}\right)^{(q-k)}}{\Delta t^q} \cdot e^{\alpha t} = \lim_{\Delta t \to 0} \frac{\left(e^{\alpha \Delta t} - 1\right)^q}{\Delta t^q} \cdot e^{\alpha t} = \alpha^q \cdot e^{\alpha t} \tag{A1.3}$$

which, for $q = 1/2$, provides the same result used in solving Eq. (2.1). The procedure shown by Eq. (A1.3) can be easily extended to any analytical function $f(t)$: the series of passages shown in the previous example defines a technique of "re-composition" of the Newton binomial expansion, that contextually *isolates* a particular *operator*, *whose* limit is finally applied to the considered function. This may be termed as an *intensive* definition because it *does not necessarily require* the convergence of the concerned expansion series. In addition, for the most habitual conditions, such a procedure is extremely simplified (see Giannantoni 2000b, 2001a).

Here it is worth pointing out *two* fundamental aspects concerning the previous definition, in particular with respect to the definition commonly used in the *traditional* Fractional Calculus:

i) the *intensive* definition here recalled presents a *multiplicity* of results similar to the case of the roots of a complex number, for example

$$\frac{d^{m/n}}{dt^{m/n}} e^{1t} = (1)^{m/n} e^{1t} = \varepsilon_i^m e^t \qquad (i = 1,2,\ldots,n) \tag{A1.4}$$

where ε_i is the *i*-th root of 1; ii) the exponential function e^t, apart from such a multiplicity of values, is an *invariant* with respect to *any order* of fractional derivation.

This latter property is the one that allows the exponential function to play that "*hinge*" role in solving *fractional* differential equations which is similar to the one that it plays in the case of ordinary differential equations. The former property, on the other hand, is the one which introduces that *variety* of solutions, typical of fractional differential equations, which enables us to analyze some well-known behavior, such as a bifurcation, in a different perspective (as shown in the text).

APPENDIX 2. "INCIPIENT" OR "SPRING" DERIVATIVE (OF INTEGER AND FRACTIONAL ORDER)

Its definition, first referred to any integer n, is given by

$$\frac{\tilde{d}^n}{\tilde{d}t^n} f(t) = \lim_{\tilde{\Delta}t:0\to 0^+} \cdot \left(\frac{\tilde{\delta}-1}{\tilde{\Delta}t} \right)^n \cdot f(t) \qquad \text{(A2.1)}$$

where the symbol $\tilde{\delta}$ represents an "operator" that *generates* a translation of a function, that is

$$\tilde{\delta}\, f(t) = f\left(t + \tilde{\Delta}t\right) \qquad \text{(A2.2)},$$

which has the following characteristics: i) the time variation $\tilde{\Delta}t$ can *also* be real, but in general it is understood as being *virtual* (and the associated symbol $\tilde{\Delta}$ reminds us of such an assumption); ii) the symbol $\tilde{\delta}\, f(t)$ is not only the representation of the second side of Eq. (A2.2), because the "operator" $\tilde{\delta}$ is *prior* with respect to $f(t)$: it is the one that *originates* such a virtual translation; iii) the "operator" $\tilde{\delta}$ may be thus better named as "generator" because, according to definition (A2.2), it "acts" as *generator* of a translation; iv) the name "generator" also reminds us that it acts in combination with something else: $\tilde{\delta}$ is in fact the *prior* "principle", $f(t)$ is the *posterior* "principle", and $f\left(t + \tilde{\Delta}t\right)$ is what "rises" from the combination of both. Such a result (or "product") is *something new*, but at the same time it retains the main genetic characteristics of its *generating* "principles".

Analogous considerations can be made with respect to the "operator" $\left(\frac{\tilde{\delta}-1}{\tilde{\Delta}t} \right)^n$.

Finally, the operation of limit ($\lim_{\tilde{\Delta}t:0 \blacklozenge 0^+}$) is here also considered as a *prior* operator with respect to those that follow it in Eq. (A2.2) but, at the same time, it is *posterior* to the very *primary* operation: the passage from the time t, initially prefixed, to the virtual time

$$\tilde{\delta}\, t = t + \tilde{\Delta}t \qquad \text{(A2.3)}$$

as a consequence of a virtual translation generated by the "generator" $\tilde{\delta}$. Such an operation is represented by the symbol $\tilde{\Delta}t : 0 \blacklozenge 0^+$ to remind us that our concept of "limit" is a "*spring-concept*": it is the "source" of what rises as a consequence of an infinitesimal virtual variation, immediately after a given time t, which in turn activates the sequence of the successive "generators" in its "spring-perspective".

Definition (A2.1) also implies

$$\frac{\tilde{d}^n}{\tilde{d}t^n}e^{\varphi(t)} = \left(\frac{\tilde{d}}{\tilde{d}t}\varphi(t)\right)^{\tilde{n}} \cdot e^{\varphi(t)} = \left(\tilde{\varphi}'\right)^n \cdot e^{\varphi(t)} \tag{A2.4}$$

so that, for *any* function $f(t) = e^{\ln f(t)}$, we consequently have

$$\frac{\tilde{d}^n}{\tilde{d}t^n} f(t) = \frac{\tilde{d}^n}{\tilde{d}t^n} e^{\ln f(t)} = \left(\frac{\tilde{f}'(t)}{f(t)}\right)^{\tilde{n}} \cdot e^{\ln f(t)} = \left(\frac{\tilde{f}'(t)}{f(t)}\right)^{\tilde{n}} \cdot f(t) = \left(\tilde{\beta}_f(t)\right)^n \cdot f(t) \tag{A2.5},$$

where the *factor of similarity* $\tilde{\beta}_f(t)$ (generally depending on time) acts as a new "generator" in Eq. (A2.5), similarly to $\tilde{\varphi}'(t)$ in Eq. (A2.4). In general it can be either a scalar quantity or a vector or even a matrix. This evidently depends on the function under consideration.

Eq. (A2.5) can be easily extended to any $q \sqcup Q$ and applied to the most common functions in Mathematical Analysis (such as, for instance, analytical functions).